On the Uniqueness of the Twisted Representation in the $\mathbb{Z}_2$ Orbifold Construction of a Conformal Field Theory from a Lattice

P.S. Montague

Department of Physics and Mathematical Physics

University of Adelaide

Adelaide

South Australia 5005

Australia

February 1, 2008

Abstract

Following on from recent work describing the representation content of a meromorphic bosonic conformal field theory in terms of a certain state inside the theory corresponding to a fixed state in the representation, and using work of Zhu on a correspondence between the representations of the conformal field theory and representations of a particular associative algebra constructed from it, we construct a general solution for the state defining the representation and identify the further restrictions on it necessary for it to correspond to a ground state in the representation space. We then use this general theory to analyze the representations of the Heisenberg algebra and its $\mathbb{Z}_2$-projection. The conjectured uniqueness of the twisted representation is shown explicitly, and we extend our considerations to the reflection-twisted FKS construction of a conformal field theory from a lattice.

1 Introduction

In [25] an argument was given for what we term the uniqueness of the twisted representation of the reflection-twisted projection of an FKS lattice conformal field theory. In this paper, we shall present an alternative and more explicit proof, as well as introducing a method which is of more general applicability and interest in its own right.

Our original motivation for proving the above-mentioned uniqueness was to enable the completion of the argument of [7, 9] which extended the "triality" of Frenkel, Lepowsky and Meurman [15] to a more general class of theories than just the Moonshine module for the Monster. While this is of sufficient import, further motivation (beyond the obvious intrinsic interest of the problem) exists. For example, this work and its generalization to higher order twists is clearly of relevance to orbifold constructions of conformal field theories in which there has been much interest as providing constructions of more physically realistic string models

1

[4, 5, 6, 2, 3, 18, 32]. It is also hoped that these and similar uniqueness arguments may be used to help complete the classification of the self-dual theories at $c = 24$ [26], *i.e.* in verifying that any conformal field theory corresponding to one of the algebras listed by Schellekens in [28, 29] is unique and further it is hoped that the abstract techniques developed here for finding representations may help in the construction of orbifolds corresponding to these algebras in the first place. In addition, this viewpoint may provide a deeper understanding of what precisely is meant by the dual of a conformal field theory and the concept of self-duality [17].

It has long been suspected that the known meromorphic representations for the reflection-twisted [8, 21] projection of an FKS (or "untwisted" [16, 30]) lattice conformal field theory are complete. We provide details later in the paper of the precise structure of these objects. We merely note here that the known representations comprise those trivially inherited from the unprojected untwisted lattice theory together with essentially (*i.e.* up to inequivalent ground states) two "twisted" representations. Since the twisted representations are both projections of a single non-meromorphic representation of the unprojected untwisted theory, we refer to this conjecture as the "uniqueness of the twisted representation".

Dong [10] has proven the result in the specific case in which the underlying lattice is taken to be the Leech lattice. This argument appears difficult to generalize however.

In [25] we found a description of a representation of a conformal field theory $\mathcal{H}$ in terms of some state P in the theory corresponding to a particular (fixed) state in the representation space. We then in some sense inverted the argument to construct a representation from P of a larger conformal field theory in which $\mathcal{H}$ is embedded as a sub-conformal field theory [23]. This allowed us to extend a representation of the $\mathbb{Z}_2$-projected theory to one of the original unprojected FKS lattice conformal field theory. (Similar ideas appear independently in [13].) The representations of this are well known [11] and the required uniqueness follows. However, several crucial technical points are ignored. For example it is not clear that the matrix elements of the representation defined in terms of P (see section 2 for more details) have the required analytic properties for the larger theory, or indeed that the Hilbert space for the representation induced by these matrix elements is even a Hilbert space. For these reasons we seek a more direct proof of the uniqueness of the twisted representation. We can thus really, in the following, only treat the conditions on P derived in [25] as necessary and not sufficient for the existence of a representation. Even in the absence of such problems with the extension (as in the "induced modules" of [13]), the following explicit analysis of the representations of the $\mathbb{Z}_2$-projected theory is clearly of general applicability and interest in its own right, and is a step towards a full understanding of the origin and nature of the twisted structure.

The layout of the paper is as follows. In section 2 we summarize the results of [25] on the description of a representation of a conformal field theory in terms of some state in the theory corresponding to a particular (fixed) state in the representation space. Then in section 3 we develop these results further, particularly in the light of work of Zhu [34], and produce a general solution to the equations in [25] as well as necessary and sufficient conditions for the state to correspond to a ground state in the representation space (*i.e.* reducing the vast degeneracy of solutions – which we must do if we are to have any hope of using this technique to classify possible representations). In section 4 we look at a simple application, namely the one-dimensional Heisenberg algebra, before attempting an analysis of the representations of the $\mathbb{Z}_2$-projection of the Heisenberg algebra (projection onto an even number of oscillators)

in section 5 and then an analysis considering also the existence of non-zero momentum eigenstates. This is finally extended to a consideration of the reflection-twisted projection of the FKS theory. We end in section 6 with some general comments and some speculations about applications to higher order twisted modules and the general construction of orbifolds ¿from conformal field theories.

2   Representations of Conformal Field Theories

We summarize here the main results of [25].

Let us first establish our notation. We define a conformal field theory (strictly a bosonic, hermitian, meromorphic conformal field theory) to consist of a Hilbert space $\mathcal{H}$, two fixed states $|0\rangle$, $\psi_L \in \mathcal{H}$, and a set $\mathcal{V}$ of "vertex operators", $i.e.$ linear operators $V(\psi, z) : \mathcal{H} \to \mathcal{H}$, $\psi \in \mathcal{H}$ parameterized by a complex parameter z such that $V(\psi_1, z_1)V(\psi_2, z_2) \cdots$ makes sense for $|z_1| > |z_2| > \cdots$,

$$V(\psi, z)|0\rangle = e^{zL_{-1}}\psi \,, \tag{1}$$

$$V(\psi, z)V(\phi, w) = V(\phi, w)V(\psi, z) \tag{2}$$

(in the sense that appropriate analytic continuations of matrix elements of either side agree) and

$$V(\psi_L, z) \equiv \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \,,, \tag{3}$$

where

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m,-n} \,, \tag{4}$$

(the constant c is the "central charge" of the theory). See [7] for a full discussion of this definition (as well as the more technical axioms omitted here).

We define a representation of this theory to consist of a Hilbert space $\mathcal{K}$ and a set of linear (vertex) operators $U(\psi, z) : \mathcal{K} \to \mathcal{K}$, $\psi \in \mathcal{H}$, such that

$$U(\psi, z)U(\phi, w) = U(V(\psi, z - w)\phi, w) \,, \tag{5}$$

and also $U(|0\rangle) \equiv 1$. Note that a relation identical to (5) is satisfied by the V's as a consequence of the above axioms [17, 7]. This representation is said to be meromorphic if matrix elements of the U's are meromorphic functions of the complex arguments.

In [25] we showed that, given a fixed quasi-primary state χ in the representation, there is some state $P(z)$ in $\mathcal{H}$ such that

$$\langle \chi|U(\psi_1, z_1)U(\psi_2, z_2) \quad \ldots \quad U(\psi_n, z_n)|\chi\rangle =$$
$$\langle P(z_n^*)|V(\psi_1, z_1 - z_n)V(\psi_2, z_2 - z_n) \quad \ldots \quad V(\psi_{n-1}, z_{n-1} - z_n)|\psi_n\rangle \,, \tag{6}$$

[Note that the definition of P given in [25] rather assumes an orbifold-like structure, and instead we might define P by

$$\langle P(z^*)|\psi\rangle = \langle \chi|U(\psi, z)|\chi\rangle \,, \tag{7}$$

for all $\psi \in \mathcal{H}$. This is essentially just taking the projection onto $\mathcal{H}$ of the definition in [25]. We shall see examples of this distinction later.] We derived necessary (and we believe

also sufficient) conditions on $P(z) \equiv \sum_{n \in \mathbb{Z}_+} P_n z^{-n}$, P_n of conformal weight n, in order that these matrix elements be the matrix elements of a representation. These are (for a *real* representation [25, 7])

$$
\begin{aligned}
\langle 0 | P(z) \rangle &= 1 \\
\overline{P(z)} &= P(-z^*) \\
e^{(z-w)L_1} P(w) &= P(z) \\
e^{zL_{-1}} P(-z) &= P(z),
\end{aligned}
\tag{8}
$$

where $\psi \mapsto \overline{\psi}$ is an antilinear map on $\mathcal{H}$ corresponding to hermitian conjugation [7].

3 Solutions for $|P\rangle$ Corresponding to a Ground State

Note that the previous section involved an arbitrary choice of state χ (in fact χ is constrained to be of unit norm and real, *i.e.* $\overline{\chi} = \chi$) in the Hilbert space for the representation. Thus any solution to the equations (8) will reflect this, and we expect an infinite number of solutions. Clearly this is of no use if we wish to use this to try to restrict the number of possible representations. In this section, we impose the further condition that χ lies in the ground state of the representation module.

The work of Zhu [34] appears to pursue many of the same ideas put forward in [25]. In particular, Zhu develops a 1-1 correspondence between representations of the conformal field theory $\mathcal{H}$ and representations of an associative algebra which he terms $A(\mathcal{H})$ (which thus in some way corresponds to the object $P(z)$ above, though at the moment the exact correspondence is unclear). In the course of this, he defines a bilinear operation $*$ on $\mathcal{H}$ and a two-sided ideal for $*$ denoted $O(\mathcal{H})$ by (rewriting his definitions in terms of the explicit modes of the vertex operators)

$$
\begin{aligned}
\psi * \phi &= \sum_{r=0}^{h} \binom{h}{r} V(\psi)_{-r} \phi \\
O(\mathcal{H}) &= \text{span} \, \{ O(\psi, \phi) \} \\
O(\psi, \phi) &= \sum_{r=0}^{h} \binom{h}{r} V(\psi)_{-r-1} \phi,
\end{aligned}
\tag{9}
$$

where ψ is of conformal weight (L_0 eigenvalue) h. Further, he shows that for $\psi \in O(\mathcal{H})$ and χ in the ground state of a representation of $\mathcal{H}$ with vertex operators U, $U(\psi)_0 \chi = 0$. Also we have the useful result

$$
U(a * b)_0 \chi = U(a)_0 U(b)_0 \chi.
\tag{10}
$$

We now apply these results in the context of our notation.

3.1 Restriction of χ to a ground state

Define $P = P(1)$ (*i.e.* $P(z) = z^{-L_0} P$). Then note from (6) that

$$
\langle \chi | U(\psi)_0 | \chi \rangle = \langle P | \psi \rangle.
\tag{11}
$$

Hence, for $\psi \in O(\mathcal{H})$,

$$\langle P|\psi\rangle = 0\,. \tag{12}$$

This condition is also sufficient for the state χ to which P corresponds to lie in the ground state. To see this, we first note from Zhu that

$$O(a,b) = c(a) * b\,, \tag{13}$$

where

$$c(a) \equiv (L_{-1} + L_0)a\,. \tag{14}$$

Following a calculation of Zhu (pages 15-16 of [34]), we see that

$$\langle\chi|U(c(a) * b)|\chi\rangle = \langle\chi|\sum_{i\in\mathbb{N}}\left(U_{-i}(c(a))U_i(b) + U_{-i-1}(b)U_{i+1}(c(a))\right)|\chi\rangle = 0\,. \tag{15}$$

But we trivially have

$$U_i(L_{-1}a) = -(i + \operatorname{wt} a)U_i(a)\,, \tag{16}$$

and thus we find that orthogonality of P to $O(\mathcal{H})$ is equivalent to

$$\langle\chi|\sum_{i\in\mathbb{N}} i\left[U_{-i}(a)U_i(b) - U_{-i}(b)U_i(a)\right]|\chi\rangle = 0\,, \tag{17}$$

for all states a and b (extending a to an arbitrary state by linearity).

In particular, taking $b = c(\overline{a})$ for a quasi-primary and using (16) again, we get a sum of norms, and hence deduce that $U_i(a)|\chi\rangle = 0$ for all quasi-primary a and all $i > 0$. The statement for all states follows simply from (16), and thus we see that χ is a highest weight state, as required.

Thus, our goal now is to solve the equations (8) for P and impose orthogonality to $O(\mathcal{H})$. This should now provide a finite set of solutions. Because of our reluctance to conclude that these conditions are also sufficient, we cannot say that each solution corresponds to a representation, but if we can find a representation corresponding to each possible solution then the classification of the possible representations will be complete.

3.2 General solution for P

We see that the first two equations in (8) are satisfied trivially by taking

$$P = |0\rangle + \sum_{\psi\text{real}} \alpha_\psi\psi\,, \tag{18}$$

where the sum is taken over all real states (of strictly positive conformal weight) and the coefficients α_ψ are real multiples of $i^{\operatorname{wt}\psi}$.

The third equation we see is equivalent to

$$(L_1 + L_0)P = 0\,. \tag{19}$$

The most general solution to this is to write

$$P = |0\rangle + \sum_{i,n} \beta_{i,n}L_{-1}{}^n\phi_i\,, \tag{20}$$

where the ϕ_i are the real quasi-primary states in $\mathcal{H}$ and $\beta_{i,n}$ satisfy

$$\beta_{i,n+1} = -\frac{1}{2}\frac{(n+h_i)\beta_{i,n}}{(n+1)h_i + \frac{1}{2}n(n+1)} , \tag{21}$$

where h_i is the conformal weight of ϕ_i. The solution to this is

$$\beta_{i,n} = \beta(h_i, n)\beta_i , \tag{22}$$

for arbitrary β_i, where

$$\beta(h, n) = \frac{(-1)^n(2h-1)!(h+n-1)!}{(h-1)!n!(2h+n-1)!} . \tag{23}$$

In order to satisfy the final relation of (8), we are required to show that $e^{\frac{1}{2}x}F_h(x)$ is an even function, where

$$F_h(x) = \sum_{n=0}^{\infty} \beta(h, n)x^{n+h} . \tag{24}$$

Set

$$\alpha_h(x) = x^{2h-1}\frac{(h-1)!}{(h-1)!}F_h(x) . \tag{25}$$

Then we have

$$\alpha_h{}^{(h)}(x) = x^{h-1}e^{-x} , \tag{26}$$

(the superscript clearly denoting repeated differentiation). Clearly we obtain (up to terms involving constants of integration)

$$\alpha_h{}^{(h-1)}(x) = -\sum_{r=0}^{h-1}\frac{(h-1)!}{(h-1-r)!}x^{h-1-r}e^{-x} . \tag{27}$$

Proceeding in this way, we find

$$\alpha_h{}^{(h-p)}(x) = (-1)^p\sum_{r_1=0}^{h-1}\sum_{r_2=0}^{h-1-r_1}\cdots\sum_{r_p=0}^{h-1-r_1-\cdots-r_{p-1}}\frac{(h-1)!}{(h-1-r_1-\cdots-r_p)!}x^{h-1-r_1-\cdots-r_p}e^{-x} , \tag{28}$$

or

$$\alpha_h{}^{(h-p)}(x) = (-1)^p\sum_{r=0}^{h-1}(-1)^r\binom{-p}{r}\frac{(h-1)!}{(h-1-r)!}x^{h-1-r}e^{-x} . \tag{29}$$

We thus find that, for some constants $q_{h,r}$,

$$\alpha_h(x) = (-1)^h\sum_{r=0}^{h-1}(-1)^r\binom{-h}{r}\frac{(h-1)!}{(h-1-r)!}x^{h-1-r}\left(e^{-x} - q_{h,r}\right) . \tag{30}$$

We claim that setting $q_{h,r} = (-1)^{h-1-r}$ forces the coefficients of x^r for $0 \leq r \leq h-1$ to vanish (which we certainly require). This is easily seen by differentiating the contribution from the integraion constants in (30) and comparing with (29), and hence $e^{\frac{1}{2}x}\alpha_h(x)$ is an odd function. Therefore, $e^{\frac{1}{2}x}F_h(x)$ is even if and only if h is even.

Thus, the general solution for $|P\rangle$ when χ is not constrained to be a ground state is

$$|P\rangle = |0\rangle + \sum_{\phi_i,n} \beta_{i,n} L_{-1}{}^n \phi_i \,, \tag{31}$$

where $\beta_{i,n}$ are as given above (with arbitrary real β_i) and the sum is over all real quasi-primary fields ϕ_i of even weight h_i.

We are left with imposing orthogonality to $O(\mathcal{H})$. However, note that this has already been achieved for much of $O(\mathcal{H})$, since

$$O(a, |0\rangle) = (L_{-1} + L_0)a \,, \tag{32}$$

which is automatically orthogonal to P by (19) ($L_n{}^\dagger = L_{-n}$). We shall impose orthogonality in the form

$$\sum_{r=0}^{h} \binom{h}{r} V(a)_{r+1} P = 0 \,, \tag{33}$$

where a is of conformal weight h. (33) must be satisfied for all $a \in \mathcal{H}$. The above comment simply becomes the observation that the term proportional to the vacuum $|0\rangle$ in (33) vanishes.

We end this section with a few simple observations. The trivial solution $P = |0\rangle$ to the above equations corresponds to the adjoint representation. When the state χ is chosen to be an eigenstate of L_0, its conformal weight is given by

$$\langle P | \psi_L \rangle \,, \tag{34}$$

¿from (11). We also note that given solutions P_1 and P_2 to the above equations, then $\alpha P_1 + (1-\alpha)P_2$ for all real α is also a solution. It corresponds to the direct sum of the corresponding representations. Also, the tensor product of solutions P_1 and P_2 corresponding to (not necessarily identical) conformal field theories $\mathcal{H}_1$ and $\mathcal{H}_2$ corresponds to representations of $\mathcal{H}_1 \oplus \mathcal{H}_2$.

4 An Application to the Heisenberg Algebra

Let us introduce some notation. We follow [8] and define the FKS (untwisted) lattice conformal field theory as follows. Let Λ be an even Euclidean lattice of dimension d. Introduce bosonic creation and annihilation operators a_n^i, $1 \le i \le d$, $n \in \mathbb{Z}$, such that

$$[a_m^i, a_n^j] = m\delta^{ij}\delta_{m,-n} \,, \qquad a_m^i{}^\dagger = a_{-m}^i \,. \tag{35}$$

Set $a_0^i \equiv p^i$ and define q^i by $[q^i, p^j] = i\delta^{ij}$. The Hilbert space is built up by the action of the a_{-n}^i, $n > 0$, on momentum states $|\lambda\rangle$, $\lambda \in \Lambda$, such that

$$p^i|\lambda\rangle = \lambda^i|\lambda\rangle \,, \qquad a_n^i|\lambda\rangle = 0 \,, \tag{36}$$

for $n > 0$.

The vertex operators are given by

$$V(\phi, z) =: \prod_{a=1}^{M} \frac{i}{(n_a - 1)!} \frac{d^{n_a}}{dz^{n_a}} X^{i_a}(x) e^{i\lambda \cdot X(z)} : \sigma_\lambda \,, \tag{37}$$

where

$$\phi = \prod_{a=1}^{M} a_{-n_a}^{i_a} |\lambda\rangle \,, \tag{38}$$

$$X^i(z) = q^i - ip^i \ln z + i \sum_{n\neq 0} \frac{a_n}{n} z^{-n} \tag{39}$$

and

$$\hat{\sigma}_\lambda \equiv e^{i\lambda\cdot q}\sigma_\lambda = \sum_{\mu\in\Lambda} \epsilon(\lambda,\mu)|\mu+\lambda\rangle\langle\mu| \,, \tag{40}$$

with

$$\hat{\sigma}_\lambda\hat{\sigma}_\mu = (-1)^{\lambda\cdot\mu}\hat{\sigma}_\mu\hat{\sigma}_\lambda \tag{41}$$

and $\epsilon(\lambda,\mu) = \pm 1$ suitably chosen [8].

This defines a conformal field theory $\mathcal{H}(\Lambda)$ of central charge d ($\psi_L = \frac{1}{2}a_{-1}\cdot a_{-1}|0\rangle$), with conjugation map given by

$$\overline{\phi} = (-1)^{L_0}\theta\phi \,, \tag{42}$$

for ϕ as in (38), with

$$\theta a_n^i \theta^{-1} = -a_n^i \,, \qquad \theta|\lambda\rangle = |-\lambda\rangle \,. \tag{43}$$

The meromorphic representations of this are known [7, 11]. They are simply given by the above vertex operators acting on the Hilbert space of states generated by the bosonic creation operators acting on momentum states $|\mu\rangle$, $\mu \in \lambda_0 + \Lambda$, for λ_0 some fixed element of Λ^*. [We must extend the definition of the $\epsilon(\lambda,\mu)$ to $\mu \in \lambda_0 + \Lambda$.] That is, we obtain $|\Lambda^*/\Lambda|$ representations (and hence a unique meromorphic representation in the case that Λ is self-dual, i.e. the conformal field theory is "self-dual" – see [25, 17] for a further discussion of this).

Now consider just the Heisenberg algebra H (i.e. $d = 1$ and we set all momenta to zero). The meromorphic representations then have Hilbert spaces built up by the action of a single set of bosonic creation operators on one-dimensional momentum states $|\mu\rangle$ for μ an arbitrary real number. Since we require χ to be real we shall take $\chi = i^{\frac{1}{2}\mu^2}|\mu\rangle_+/\sqrt{2} \equiv i^{\frac{1}{2}\mu^2}(|\mu\rangle + |-\mu\rangle)/\sqrt{2}$ or $\chi = ii^{\frac{1}{2}\mu^2}|\mu\rangle_-/\sqrt{2} \equiv i^{\frac{1}{2}\mu^2}i(|\mu\rangle - |-\mu\rangle)/\sqrt{2}$. The corresponding P is then, in both cases, found to be [25, 8]

$$\overline{W}(\chi,1)e^{L_1}|\chi\rangle = \cosh{(\mu X_-)}|0\rangle \equiv \mathrm{P}_\mu \,, \tag{44}$$

where $X_- = \sum_{n=1}^\infty (-1)^n \frac{a_{-n}}{n}$. (We ignore terms in the sectors with momentum $\pm 2\mu$, since they will give no contribution to the matrix element in (6). Note that this is an example illustrating the comment on the definition of P following equation (6).) Let us try to derive this result from our approach.

It turns out that in this simple case we do not need the general solution for P derived in the last section. We simply impose orthogonality to $O(H)$. It is easily shown (Lemma 2.1.2 of [34]) that $(a_{-n} + a_{-n-1})|0\rangle \in O(H)$, and we deduce that the most general solution for P orthogonal to $O(H)$ and satisfying the first two relations of (8) is

$$P = |0\rangle + \sum_{n=1}^\infty \lambda_n X_-{}^n|0\rangle \,, \tag{45}$$

where the λ_n are arbitrary real coefficients. Trivially from the general solution (31) for P, $\lambda_n = 0$ for n odd. Since $L_{\pm 1}$ do not mix states with different numbers of creation operators (in the zero-momentum sector) we see that we can get no further information from this and that the solution we have is complete (equivalently we could observe that since P_μ satisfies the constraint equations for all μ then we can see that (45) (with $\lambda_n = 0$ for n odd) must do so also).

To obtain the required form for P, we could impose that χ be an L_0 eigenstate by setting

$$\langle \chi | L_0{}^n | \chi \rangle = \langle \chi | L_0 | \chi \rangle^n . \tag{46}$$

But

$$\begin{aligned}
\langle \chi | L_0{}^n | \chi \rangle &= \langle \chi | U(\psi_L)_0 \ldots U(\psi_L)_0 | \chi \rangle \\
&= \langle \chi | U(\psi_L * \cdots * \psi_L)_0 | \chi \rangle \\
&= \langle P | \psi_L * \cdots * \psi_L \rangle ,
\end{aligned} \tag{47}$$

¿from (11) and (10), and so we get

$$\langle P | \psi_L \rangle^n = \langle P | \psi_L * \cdots * \psi_L \rangle . \tag{48}$$

This clearly fixes λ_{2n} in terms of the lower order coefficients, and so we see from the known solution that we must obtain $\lambda_{2n} = \frac{\lambda^{2n}}{(2n)!}$ for some real λ.

In this case, we find that all solutions to our constraint equations correspond to representations (reinforcing our belief that the conditions are actually sufficient). Also note that, though there was no a priori imposition that the representation be meromorphic (and non-meromorphic representations certainly exist, as we shall see in the next section), orthogonality to $O(H)$ seems to have restricted to meromorphic representations (some of the manipulations we used are strictly only valid in this case, so this is not too surprising). It is also worth noting that the representation we obtain is not irreducible, since it contains both that built up from $|\lambda\rangle$ and that ¿from $|-\lambda\rangle$, though the distinct solutions we have found do correspond to inequivalent representations as can be trivially seen by the fact that the conformal weight of the ground state is distinct. In general though, we will have to do more work after solving our equations to identify the inequivalent as well as the irreducible representations of a given theory.

5 The $\mathbb{z}_2$-Twisted Heisenberg Algebra and the Reflection-Twisted FKS Lattice Conformal Field Theory

5.1 Notation and known results

We begin by introducing a non-meromorphic representation of the conformal field theory $\mathcal{H}(\Lambda)$ (see [8] for details). We start with an irreducible representation of the gamma matrix algebra

$$\gamma_\lambda \gamma_\mu = (-1)^{\lambda \cdot \mu} \gamma_\mu \gamma_\lambda , \tag{49}$$

c.f. (41). We denote a typical state in such a representation by ρ. The Hilbert space for our representation of $\mathcal{H}(\Lambda)$ is built up from this space by the action of bosonic creation

oscillators with non-integral grading, *i.e.* we introduce operators c_r^i, $1 \leq i \leq d$, $r \in \mathbb{Z} + \frac{1}{2}$, such that

$$[c_r^i, c_s^j] = r\delta^{ij}\delta_{r,-s}\,, \qquad c_r^{i\,\dagger} = c_{-r}^i\,, \tag{50}$$

and

$$c_r^i\rho = 0 \tag{51}$$

for $r > 0$. The "twisted" vertex operator (the U of our general theory) for ϕ as in (38) is given by [8]

$$V_T(\phi, z) = V_0\left(e^{\Delta(z)}\phi, z\right)\,, \tag{52}$$

where

$$V_0(\phi, z) =: \prod_{a=1}^M \frac{i}{(n_a - 1)!}\frac{d^{n_a}}{dz^{n_a}}R^{i_a}(x)e^{i\lambda \cdot R(z)} : \gamma_\lambda\,, \tag{53}$$

with

$$R^i(z) = i\sum_{r \in \mathbb{Z} + \frac{1}{2}}\frac{c_r}{r}z^{-r} \tag{54}$$

and

$$\Delta(z) = -\frac{1}{2}p^2 \ln 4z + \frac{1}{2}\sum_{\substack{m,n \geq 0 \\ (m,n) \neq (0,0)}}\binom{-\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n}\frac{z^{-m-n}}{m+n}a_m \cdot a_n\,. \tag{55}$$

This is found to define a representation of $\mathcal{H}(\Lambda)$, which we denote $\mathcal{H}_T(\Lambda)$, the so-called $\mathbb{Z}_2$-twisted representation. Note that it is non-meromorphic however. The matrix elements of the V_T's contain square root branch cuts in general. Also note that the conformal weight of the ground state of the representation is found to be $d/16$. The involution θ defined on $\mathcal{H}(\Lambda)$ can be lifted to this representation by

$$\theta\rho = \rho\,, \qquad \theta c_r^i\theta^{-1} = -c_r^i\,. \tag{56}$$

[Note that this is gives an involution on the representation when d is a multiple of 8 if we define instead $\theta\rho = (-1)^{d/8}\rho$.]

We set

$$\mathcal{H}_T(\Lambda)_\pm = \{\zeta \in \mathcal{H}_T(\Lambda)\ :\ \theta\zeta = \pm\zeta\}\,, \tag{57}$$

with a similar decomposition for $\mathcal{H}(\Lambda)$. Then $\mathcal{H}(\Lambda)_\pm$ and $\mathcal{H}_T(\Lambda)_\pm$ are found to form meromorphic irreducible representations of $\mathcal{H}(\Lambda)_+$ [7]. Our main result will be that these are the only such representations, at least for Λ self-dual.

Let us now, as before, restrict to one dimension and set all momenta to zero, *i.e.* we will study representations of H_+, the $\theta = 1$ projection of the Heisenberg algebra H.

Let us look at the known results before we start to analyze solutions of our equations. From the above, we have meromorphic representations corresponding to real ground states (*i.e.* possible choices for χ),

$$\chi_1 = i^{\frac{1}{2}\lambda^2}|\lambda\rangle_+\,, \quad \chi_2 = ii^{\frac{1}{2}\lambda^2}|\lambda\rangle_-\,, \quad \chi_3 = a_{-1}|0\rangle\,,$$
$$\chi_4 = \rho\,, \qquad \chi_5 = c_{-\frac{1}{2}}^i\rho'\,, \tag{58}$$

for suitable ρ and ρ' (see [8] for a discussion of the action of conjugation on the twisted sector ground states). (The term χ_3 arises since $|0\rangle_- = 0$.) We now calculate the corresponding

P's. As before, χ_1 and χ_2 give rise to P_λ (44), again ignoring contributions to P with non-zero momentum. The other three χ's give us three exceptional solutions P_a, P_b and P_c corresponding to ground states of conformal weight 1, $\frac{1}{16}$ and $\frac{9}{16}$ respectively. We evaluate them to be

$$
\begin{aligned}
P_a &= |0\rangle - \sum_{n=1}^{\infty}(-1)^n a_{-1}a_{-n}|0\rangle \\
P_b &= e^{\Delta(1)^\dagger}|0\rangle \\
P_c &= \left(1 - 2\sum_{n,m>0}\binom{\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n}a^i_{-m}a^i_{-n}\right)e^{\Delta(1)^\dagger}|0\rangle \,.
\end{aligned}
\tag{59}
$$

5.2   Solution in a simple case

Let us start with a simple exercise to see how these solutions corresponding to particular values of the conformal weight of the representation ground state might arise.

We shall look for solutions for P corresponding to representations of H_+ in which P has no more than two creation operators in any one term. From the above results and our conjecture of their completeness, we would expect only to obtain the trivial solution $|0\rangle$ and P_a.

We first write down a general solution to the constraint equations (8) using (31), and then impose orthogonality to $O(H_+)$. The quasi-primary states in H_+ containing two oscillators are easily found. They are, at even levels $2h$,

$$
\phi_h = \sum_{r=1}^{h}\frac{(-1)^r}{r}\binom{2h-1}{r-1}P_{r,2h-r} - \frac{1}{2}\frac{(-1)^h}{h}\binom{2h-1}{h-1}P_{h,h}\,,
\tag{60}
$$

where $P_{n_1 n_2 \ldots} \equiv a_{-n_1}a_{-n_2}\cdots|0\rangle$, and so we write

$$
P = |0\rangle + \sum_{h=1}^{\infty}\sum_{n=0}^{\infty}\beta(2h,n)\alpha_h L_{-1}{}^n\phi_h\,,
\tag{61}
$$

for some real coefficients α_h. We now impose (48), *i.e.* we take χ to be a conformal eigenstate. This clearly gives us α_h in terms of α_1. Set $\alpha_1 = -2\gamma$ (γ will be the conformal weight of the representation (from (34)). Explicitly we find

$$
\begin{aligned}
\alpha_2 &= -\frac{1}{3}\gamma\left(\gamma + \frac{1}{5}\right) \\
\alpha_3 &= -\frac{1}{45}\gamma\left(\gamma^2 + \gamma + \frac{1}{7}\right),
\end{aligned}
\tag{62}
$$

and, up to and including states of conformal weight 6,

$$
\begin{aligned}
P &= |0\rangle + \gamma P_{11} - \gamma P_{12} + \frac{1}{3}\gamma(\gamma+2)P_{13} - \frac{1}{4}\gamma(\gamma-1)P_{22} - \frac{1}{3}\gamma(\gamma-1)P_{23} - \\
&\quad \frac{1}{2}\gamma(\gamma+1)P_{14} + \frac{\gamma}{45}\left(\gamma^2 + 26\gamma + 18\right)P_{15} - \frac{\gamma}{36}\left(2\gamma^2 + 7\gamma - 9\right)P_{24} + \\
&\quad \frac{\gamma}{27}\left(\gamma^2 - 4\gamma + 3\right)P_{33}\,.
\end{aligned}
\tag{63}
$$

11

Now, ignoring terms with more than two oscillators,

$$O(P_{11}, P_{12}) = 2P_{22} + 4P_{13} + 4P_{23} + 8P_{14} + 2P_{24} + 4P_{15} \,, \qquad (64)$$

and so setting this orthogonal to P we find

$$\gamma^3 + 35\gamma^2 - 36\gamma = 0 \,. \qquad (65)$$

This has roots $\gamma = 0, 1$ as required, as well as the root $\gamma = -36$ which we can dismiss by unitarity (though it is spurious and we will see from the argument that we use in the general case that $\gamma = 0, 1$ are the only possible solutions). $\gamma = 0$ gives $P = |0\rangle$, while $\gamma = 1$ must give $P = P_a$, since this solution must arise from this analysis (we evaluate the first few terms as a check on our techniques).

5.3   General solution for H_+

Let us now attempt the general case. Instead of imposing (48), we shall require the (more restrictive) condition, using (10), $\langle P | \psi_L * \phi \rangle = \gamma \langle P | \phi \rangle$ for all $\phi \in H_+$ and with γ the required conformal weight of the ground state (the coefficient of P_{11} in P in this case). Since ϕ is arbitrary, this condition amounts to requiring

$$(L_2 - L_0)P = \gamma P \,, \qquad (66)$$

using (19) and the definition of $*$ in (9). This is easier to use in practise than (48), and it turns out is also more restrictive (since we are using (10), which itself depends on χ being a ground state in the representation). Note that (66) together with (31) will give us a solution for P up to arbitrary coefficients for the primary states at even levels. Since there are no primary states involving just two oscillators, then P in the above simple example would be determined completely in terms of γ (so we see immediately that (66) is strictly more powerful than (48)). However, in the general case, an infinite number of primary states occur and we obtain a corresponding set of unknown coefficients which must be constrained by imposing (33).

   We know from the Kac determinant [19] that the even levels at which primary states occur are given by $4n^2$, $n = 0, 1, \ldots$. So, we expect to obtain at least one new parameter at level 4 (in fact exactly one, but we shall postpone a discussion of the explicit structure of the primary states until it is needed later in the argument). We start from (31), and so must consider the quasi-primary states. Since L_1 does not mix terms with different numbers of oscillators, we can consider quasi-primary states at a given level and with a certain number of oscillators. The two oscillator ones ϕ_h at level $2h$ are as given above. We denote their coefficients in the expansion (31) of P as γ_h, and set as before $\gamma_1 = -2\gamma$, γ the conformal weight of the ground state of the representation. We find a 4-oscillator quasi-primary state at level 4, which is simply P_{1111}. We denote the coefficient of this in the expansion (31) of P as ρ. At level 6 we have, in addition to the quasi-primary state with two oscillators, one 4-oscillator one

$$3P_{1122} - 4P_{1113} \,, \qquad (67)$$

whose coefficient we denote by δ, and one trivial 6-oscillator state P_{111111}.

Imposing (66), we find

$$
\begin{aligned}
\gamma_2 &= 2\rho - \frac{1}{3}\gamma\left(\gamma + \frac{1}{5}\right) \\
45\gamma_3 &= \frac{9}{5}\rho\left(\frac{8}{3} + 4\gamma\right) - \frac{6}{5}\gamma^3 - \frac{4}{5}\gamma^2 - \frac{1}{7}\gamma \\
\delta &= -\frac{\gamma^2}{180} + \frac{\gamma^3}{180} - \frac{\rho}{45} - \frac{\gamma\rho}{30}.
\end{aligned}
\tag{68}
$$

Note that the expression for γ_3 differs from that in (62) when we take $\rho = 0$, but we are now using the more powerful relation (66) and the expressions in any case then agree for the values $\gamma = 0$, 1 which is all we can really require. Note also that if we set $\delta = \rho = 0$ (*i.e.* again restrict to at most two oscillators) then we again find $\gamma = 0$, 1 (this time without the spurious negative root).

Now that we have some experience with this technique, let us consider what exactly the result is which we are trying to find. If we are to find P_λ, P_a, P_b and P_c as the only possible solutions, then we trivially see we must have $\rho = \frac{\gamma^2}{6}$ except at $\gamma = 1$, $\frac{1}{16}$ and $\frac{9}{16}$. (We expect there to be no constraint on ρ when γ is one of the special values listed, since in that case we expect $\alpha P_\lambda + (1 - \alpha)P_x$ ($x = a$, b or c) to be a solution for all real α.) The lowest degree polynomial which will provide such a relation is of degree 5 in γ. We see, for the techniques we are using, that this can come only ¿from a state of conformal weight at least 10. Since the computation grows rapidly with increasing conformal weight, we hope that the required level is exactly 10. It would be expected that some new feature occurs at this level.

Let us look at the numbers of quasi-primary states at the various levels. From the partition function

$$
1 + x^2 + x^3 + 3x^4 + 3x^5 + 6x^6 + 7x^7 + 12x^8 + 14x^9 + 22x^{10} + \dots ,
\tag{69}
$$

we see that there are 5 quasi-primary states at level 8 and 8 at level 10. We can construct a set of quasi-primary states from the known two-oscillator ones. For example, at level 4 we can write the two quasi-primaries as ϕ_2 and $\phi_1{}^2$ (using an obvious notation – more precisely we are projecting the $*$ product $\phi_1 * \phi_1$ onto the state of highest conformal weight). We find that this gives all quasi-primary states at levels 2, 4, 6 and 8, but at level 10 we have to consider in addition the quasi-primary state $P_{1144} - 4P_{1234} + \frac{64}{27}P_{1333} + 2P_{4222} - \frac{4}{3}P_{2233}$, which is the required new feature. Imposing (66) on (31) then fixes all unknown parameters in P up to level 10 in terms of ρ and γ (since the next primary state is at level 16). We find that this imposition is consistent, and we must consider (33) if we are to get our required constraint.

The first non-trivial state to try in (33) should be P_{22}, since the action of the Virasoro algebra on P is determined by the action of L_2 and L_1 (which have been considered exhaustively), and the first state not in the Virasoro module on the vacuum is P_{22}. We find that

$$
[V(P_{22})_m, a_p] = -2p(1 + m + p)(1 - p)a_{p+m}.
\tag{70}
$$

Acting upon our expansion of P, we find that all terms vanish up to and including level 4, but at level 5 we obtain some potentially non-zero coefficients. Equating the coefficient of P_{1112} to zero gives

$$
(9 - 169\gamma + 416\gamma^2 - 256\gamma^3)\left(\rho - \frac{\gamma^2}{6}\right) = 0 ,
\tag{71}
$$

13

which is the required result.

All that remains now is to argue that the terms at higher levels in P are given uniquely in terms of those which we have already computed. This unique form must then be the same as the known solutions.

We consider the primary states of the theory. We have a specific form for these ¿from [1, 20, 33]. We find that the primary fields occur at level n^2, $n \in \mathbb{Z}$, and that the unique [20, 33] primary at level $h = n^2$ is given by

$$\underbrace{S_{n, \ldots, n}}_{n \text{ terms}} \left(\sqrt{2} a_{-j}/j \right) |0\rangle , \tag{72}$$

where the Schur polynomial associated to a partition $\lambda = \{\lambda_1 \geq \lambda_2 \geq \ldots\}$ is

$$S_{\lambda_1, \lambda_2, \ldots}(x) = \det \left(S_{\lambda_i + j - i}(x) \right)_{i,j} , \tag{73}$$

and the elementary Schur polynomials $S_k(x)$ are defined by

$$\sum_{k \geq 0} S_k(x) z^k = \exp \left(\sum_{k \geq 1} x_k z^k \right) . \tag{74}$$

Let us show, for reasons that will become clear, that the coefficient of $a_{-1}^{n^2}|0\rangle$ in the primary field is non-zero. It is easily seen to be

$$\det \left(1/(n + j - i)! \right) , \tag{75}$$

and this elementary determinant we find, for example ¿from [22], to be

$$\frac{\prod_{i=1}^{n} i!}{\prod_{i=n}^{2n-1} i!} , \tag{76}$$

giving the required result.

Rather than use the result for the primary field given in (72), we find it easier to construct the terms we require for our argument explicitly. We begin by writing down the quasi-primary states at level h with h, $h - 2$, $h - 4$ and $h - 6$ oscillators. They are found to be

$$\psi_1 = a_{-1}^{h}|0\rangle$$

$$\psi_2 = 4a_{-1}^{h-3}a_{-3}|0\rangle - 3a_{-1}^{h-4}a_{-2}^2|0\rangle$$

$$\psi_3 = 3a_{-1}^{h-8}a_{-2}^4|0\rangle + \frac{16}{3}a_{-1}^{h-6}a_{-3}^2|0\rangle - 8a_{-1}^{h-7}a_{-2}^2a_{-3}|0\rangle$$

$$\psi_4 = 2a_{-1}^{h-5}a_{-5}|0\rangle - 5a_{-1}^{h-6}a_{-2}a_{-4}|0\rangle + \frac{10}{3}a_{-1}^{h-6}a_{-3}^2|0\rangle$$

$$\psi_5 = a_{-1}^{h-12}a_{-2}^6|0\rangle - \frac{64}{27}a_{-1}^{h-9}a_{-3}^3|0\rangle + \frac{16}{3}a_{-1}^{h-10}a_{-2}^2a_{-3}^2|0\rangle - 4a_{-1}^{h-11}a_{-2}^4a_{-3}|0\rangle$$

$$\psi_6 = a_{-1}^{h-7}a_{-7}|0\rangle - \frac{7}{2}a_{-1}^{h-8}a_{-2}a_{-6}|0\rangle + 7a_{-1}^{h-8}a_{-3}a_{-5}|0\rangle - \frac{35}{8}a_{-1}^{h-8}a_{-4}^2|0\rangle$$

$$\psi_7 = a_{-1}^{h-9}a_{-2}^2a_{-5}|0\rangle - \frac{5}{3}a_{-1}^{h-9}a_{-2}a_{-3}a_{-4}|0\rangle + \frac{20}{27}a_{-1}^{h-9}a_{-3}^3|0\rangle - \frac{4}{3}a_{-1}^{h-8}a_{-3}a_{-5}|0\rangle$$

$$\qquad + \frac{5}{4}a_{-1}^{h-8}a_{-4}^2|0\rangle$$

$$\psi_8 = \frac{1}{2}a_{-1}^{h-8}a_{-4}^2|0\rangle - 2a_{-1}^{h-9}a_{-2}a_{-3}a_{-4}|0\rangle + \frac{32}{27}a_{-1}^{h-9}a_{-3}^3|0\rangle - \frac{2}{3}a_{-1}^{h-10}a_{-2}^2a_{-3}^2|0\rangle$$

$$\qquad + a_{-1}^{h-10}a_{-2}^3a_{-4}|0\rangle . \tag{77}$$

We then impose that the primary state is annihilated by L_2. This determines the coefficients ϵ_n of ψ_n in terms of ϵ_1, which we take (in the light of the above analysis) to be 1 for convenience. We find in particular

$$
\begin{aligned}
\epsilon_3 &= \frac{1}{384} h(h-1)(h-4)(h-21) \\
\epsilon_4 &= \frac{1}{20} h(h-1)(h-4) \\
\epsilon_6 &= -\frac{5}{36} h(h-1)(h-4)(h-9) \, .
\end{aligned}
\tag{78}
$$

We consider the action of $V\left(P_{22}\right)_5$ on the primary state at level $h = n^2$. If we can show that this is non-zero, then the relation (33) for $a = P_{22}$ will give the coefficient of the primary state in terms of the coefficients of states at lower conformal weight. We will consider the term $a_{-1}{}^{h-7} a_{-2}|0\rangle$ which arises as a result of the action. The relevant pieces of $V\left(a_{-2}{}^2|0\rangle\right)_5$ are

$$
24 a_2 a_3 + 20 a_1 a_4 - 16 a_{-2} a_7 \, ,
\tag{79}
$$

and we need to consider their action on the states $a_{-1}{}^{h-7} a_{-7}|0\rangle$, $a_{-1}{}^{h-7} a_{-2}{}^2 a_{-3}|0\rangle$ and $a_{-1}{}^{h-6} a_{-2} a_{-4}|0\rangle$. Assign these coefficients y_1, y_2 and y_3 respectively. We find trivially that the coefficient of $a_{-1}{}^{h-7} a_{-2}|0\rangle$ will be

$$
- 112 y_1 + 288 y_2 + 80(h-6) y_3 \, ,
\tag{80}
$$

and substituting in our results for the primary state we get

$$
- 4 h(h-1)(h-4)(4h-39) \, ,
\tag{81}
$$

which does not vanish over the range of relevant values for h.

This completes the proof that the only possible solutions for P for the algebra H_+ are P_λ, P_a, P_b and P_c. That each correspond to consistent meromorphic representations is known from the explicit results of previous work.

5.4 General solution for $\mathcal{H}_+(\Lambda)$, Λ one-dimensional

We now consider representations of the theory $\mathcal{H}_+(\Lambda)$ in the case where Λ is a one-dimensional lattice.

We already have the following $P's$ (c.f. (44) and (59))

$$
\begin{aligned}
\hat{P}_\mu^\pm &\equiv \cosh(\mu X_-)\left(|0\rangle \pm \sqrt{2}|2\mu\rangle_+\right) \\
\hat{P}_a &\equiv P_a \\
\hat{P}_b &\equiv \sum_{\lambda \in \Lambda} \rho^\dagger \gamma_\lambda \rho \, e^{\Delta(1)^\dagger} |\lambda\rangle \equiv \frac{1}{2} \sum_{\lambda \in \Lambda_0} S_\lambda e^{\Delta(1)^\dagger} |\lambda\rangle_+ \\
\hat{P}_c &\equiv \sum_{\lambda \in \Lambda} \rho^\dagger \gamma_\lambda \rho \left(1 - 2 \sum_{n,m \geq 0} \binom{\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n} a_{-m}^i a_{-n}^i \right) e^{\Delta(1)^\dagger} |\lambda\rangle \\
&\equiv \frac{1}{2} \sum_{\lambda \in \Lambda_0} S_\lambda \left(1 - 2 \sum_{n,m \geq 0} \binom{\frac{1}{2}}{m}\binom{-\frac{1}{2}}{n} a_{-m}^i a_{-n}^i \right) e^{\Delta(1)^\dagger} |\lambda\rangle_+ \, ,
\end{aligned}
\tag{82}
$$

where $\Lambda_0 = \{\lambda \in \Lambda : \frac{1}{2}\lambda^2 \text{ even}\}$, corresponding to representation states as before (*i.e.* (58)), which are easily evaluated from the known representations detailed earlier. (Note though that $|\mu\rangle_\pm$ is only a ground state for a meromorphic representation if μ is a vector of minimal norm in the cosets Λ^*/Λ, and of course the term $|2\mu\rangle_+$ is present only if $2\mu \in \Lambda$.) [In one dimension, the cocycle structure is trivial, and if we take λ_0 to be a basis vector for Λ then the only (irreducible) representations (with $\gamma_\lambda = \gamma_{-\lambda} = (-1)^{\frac{1}{2}\lambda^2}\gamma_\lambda{}^\dagger$ and also for which a suitable charge conjugation matrix exists [8]) are one or two-dimensional and are such that $S_{n\lambda_0}$ is either 1 or $(-1)^n$ in the case that $\frac{1}{2}\lambda_0{}^2$ is even, and is $-\frac{1}{2}(1 + (-1)^n)$ when $\frac{1}{2}\lambda_0{}^2$ is odd (so that $S_\lambda \equiv -1$ on Λ_0), giving us the above results.]

Let us now verify our uniqueness conjecture that these are the only solutions corresponding to meromorphic representations. We essentially follow the argument of the preceding subsection (though the algebra is considerably more intricate), and so shall simply sketch the proof.

We shall evaluate the terms in P with momentum $\pm\lambda$, $\lambda \in \Lambda$. As before, we begin by writing down the quasi-primary states at even levels and expand P as in (31). Imposing (66) then fixes, as before, all of the unknown coefficients except for those of primary states at even levels. From the known results and our uniqueness conjecture, we would expect to derive the constraint $\gamma = \frac{1}{16}$, $\gamma = \frac{9}{16}$ or $\lambda^2 = 8\gamma$ (the last of these corresponding to $\hat{P}_\mu$). Thus we expect a cubic equation in γ, and would thus naively expect to have to evaluate P as far as 6 levels above $|\lambda\rangle_+$.

We have explicit forms for the primary states, ¿from *e.g.* [1], which tells us (remembering that the momentum λ is constrained so that λ^2 is even) that the primary states are at levels $k(k+2d)$ above $|\sqrt{2}d\rangle_+$ for k and d non-negative integers. In addition, the conformal weights of the primaries must be even for an allowable contribution to P.

We demonstrate below that the contribution from a primary state at level at least 7 above $|\sqrt{2}d\rangle_+$ is determined in terms of the coefficients of the primary states at lower levels (just as we did to complete the proof in the case of zero momentum in the above subsection). This, together with the known (even) levels of the primary states is easily seen to give that for d odd and at least 3 all contributions to P vanish, while for d even everything is determined in terms of the coefficient of the primary state $|\sqrt{2}d\rangle_+$.

The case $d = 1$ must be considered separately. There is a primary state 3 levels above $|\sqrt{2}\rangle_+$ (then the next is at level 15). This state is explicitly

$$a_{-1}{}^3|\sqrt{2}\rangle_- - \frac{3}{\sqrt{2}}a_{-1}a_{-2}|\sqrt{2}\rangle_+ + a_{-3}|\sqrt{2}\rangle_- , \tag{83}$$

and a simple calculation confirms that it is not annihilated by $V(|\sqrt{2} >_+)_2$, and hence (33) shows that its coefficient in P must vanish. Then the argument below that higher primary states are determined in terms of the lower ones shows that there are no contributions to P corresponding to $d = 1$.

We now give the argument that the higher order primary states are determined as described above. As in (72-76), we find that the coefficient of the term $a_{-1}^h|\lambda\rangle_\pm$ is non-zero in the primary state at level h above the state $|\lambda = \sqrt{2}d\rangle_+$ (with $h = k(k + 2d)$). We then construct the primary state by hand in terms of states of successively decreasing numbers of oscillators, as before. We omit the details as, though the technique is straightforward, the explicit forms involved are unwieldy. As in the zero momentum case, we evaluate the action of $V(P_{22})_5$ on the primary. In particular, we consider the term $a_{-1}{}^{h-5}|\lambda\rangle_\mp$ (for which we require

the coefficients r, s and t of the states $a_{-1}{}^{h-5}a_{-2}a_{-3}|\lambda\rangle_{\mp}$, $a_{-1}{}^{h-4}a_{-4}|\lambda\rangle_{\mp}$ and $a_{-1}{}^{h-5}a_{-5}|\lambda\rangle_{\pm}$).
We find that the coefficient of $a_{-1}{}^{h-5}|\lambda\rangle_{\mp}$ is

$$144r + 80(h-4)s + 60\lambda t = 4\sqrt{2}hd(110 - 89h + 12h^2 + 157d^2 - 40hd^2 + 12d^4). \tag{84}$$

To eliminate the (presumably) spurious roots to the vanishing of this, we consider also the coefficient of the term $a_{-1}{}^{h-7}|\lambda\rangle_{\mp}$ in the action of $V(P_{33})_5$ on the primary. This turns out to be

$$72(14a + 35b + 50c + 7\lambda d), \tag{85}$$

where a, b, c and d are the coefficients of $a_{-1}{}^{h-6}a_{-6}|\lambda\rangle_{\mp}$, $a_{-1}{}^{h-7}a_{-2}a_{-5}|\lambda\rangle_{\mp}$, $a_{-1}{}^{h-7}a_{-3}a_{-4}|\lambda\rangle_{\mp}$ and $a_{-1}{}^{h-7}a_{-7}|\lambda\rangle_{\pm}$ in the primary state respectively. This we evaluate to be

$$12\sqrt{2}hd(5181 - 4737h + 885h^2 - 33h^3 + 12167d^2 - 4945hd^2 + 316h^2d^2 + 2824d^4 - 364hd^4 + 48d^6), \tag{86}$$

and eliminating h between the vanishing of (84) and (86) gives

$$2304 \quad (d-1)(d+1)(2d-3)(2d+3)(2d-1)(2d+1)(-14402773 + 38418179d^2 - 18390543d^4 + 1290780d^6) = 0, \tag{87}$$

which has no integer solutions except $d = 1$. In the case $d = 1$, the only integer solution for h to (84) is $h = 3$, which is the state we considered separately above. This completes our argument.

We may now evaluate the contributions to P for states of momentum $\pm\lambda$ ($\frac{1}{2}\lambda^2$ even) as far as level 6 (up to which there are no primary states other than $|\lambda\rangle_+$ itself). There is one quasi-primary state at level 2 above $|\lambda\rangle_+$, two at level 4 and 4 at level 6 to consider. The arguments are exactly as before. We obtain a consistent solution which we do not detail here since its exact structure is not illuminating and the expressions involved are again unwieldy. Following the preceding analysis with zero momentum, we then impose (33) with $a = P_{22}$. The coefficient of the term $a_{-1}|\lambda\rangle_-$ is found to be

$$32\lambda\frac{(\gamma - \frac{1}{16})(\gamma - \frac{9}{16})(\lambda^2 - 8\gamma)(\lambda^2 - 2)}{(2\lambda^2 - 1)^2(4\lambda^4 - 6\gamma\lambda^2 + 225)}C_\lambda, \tag{88}$$

where C_λ is the coefficient of $|\lambda\rangle_+$ in P. We thus find the required result.

All that remains to fix the form of P exactly is to evaluate the coefficients C_λ. The relation (33) with $a = |\lambda_0\rangle_+$ (λ_0 a basis vector for the lattice, as above) will give a relation between $C_{\lambda+\lambda_0}$ and $C_{\lambda-\lambda_0}$. The coefficient of $C_{\lambda+\lambda_0}$ in this will be non-zero if the difference of the conformal weights of $|\lambda\rangle_+$ and $|\lambda - \lambda_0\rangle_+$ is at least $\frac{1}{2}\lambda_0{}^2 + 1$ (the highest weight in (33)). This is true, for $\lambda = n\lambda_0$, if $n > 1$. Thus, we see that we can determine every coefficient C_λ in terms of C_0 (which is fixed equal to 1 by (8)) and C_{λ_0}. The arbitrariness of C_{λ_0} (if $\frac{1}{2}\lambda_0{}^2$ is even, *i.e.* if $\lambda_0 \in \Lambda_0$ – otherwise $C_{\lambda_0} = 0$ as the primary state $|\lambda_0\rangle_+$ is of odd conformal weight) reflects the fact that we are able, in the case of the twisted representations, to take a linear combination of known solutions with $S_{n\lambda_0} = 1$ and $S_{n\lambda_0} = (-1)^n$ (*i.e.* the solution to the equations for P does not correspond to an irreducible representation, as we have discussed before). Similarly, in the case of a representation based on a momentum state $|\mu\rangle_\pm$, we expect the coefficient of $|2\mu\rangle_+$ (when $2\mu \in \Lambda$) to be arbitrary, because we obtain an analogous linear combination of $\hat{P}_\mu^\pm$.

Note that, in order to constrain $\mu \in \Lambda^*$, we cannot use (33), but must evaluate for example the four-point function

$$\langle \mu | V(|\lambda\rangle_+, z) V(|\lambda\rangle_+, w) | \mu \rangle \tag{89}$$

This is easily done (see *e.g.* [7]), and we find terms of the form $z^{\pm\lambda\mu}$. The restriction that the representation be meromorphic gives the required constraint. Note that this is our first example of a case in which solving all of our equations for P does not automatically give a meromorphic representation.

5.5   Extension to d dimensions

In this section we shall describe the extension of the above results to the case of d dimensions. We only sketch the main arguments, since the techniques used are straightforward though messy. In any case, such arguments may clearly be made rigorous if desired.

Let us first consider the case of the $\mathbb{Z}_2$ projection H^d_+ of the Heisenberg algebra in d dimensions.

Now, in more than one dimension, it can be shown [7] that the theory H^d_+ is generated by the modes of the vertex operators corresponding to the states $a^i_{-1}a^j_{-1}|0\rangle$, $1 \le i \le j \le d$. Thus, we see that we shall only need to consider the matrix elements involving such states in order to fix the representation uniquely. In practice, it turns out to be more convenient however to take the following (though closely related) approach.

We first note that

$$\prod_{a=1}^{M} a^{i_a}_{-m_a}|0\rangle * \prod_{b=1}^{N} a^{j_b}_{-n_b}|0\rangle = \prod_{a=1}^{M} a^{i_a}_{-m_a} \prod_{b=1}^{N} a^{j_b}_{-n_b}|0\rangle + \dots , \tag{90}$$

where $\dots$ denotes terms containing less than M+N oscillators. We see that by repeating this process we may thus write any state in H^d_+ as a sum of $*$ products of states of the form $a^i_{-m}a^j_{-n}|0\rangle$. Hence we find that we need only consider matrix elements of the form

$$\langle \chi | U(a^{i_1}_{-m_1}a^{j_1}_{-n_1}|0\rangle * \dots a^{i_N}_{-m_N}a^{j_N}_{-n_N}|0\rangle)_0 |\chi\rangle = \langle \chi | U(a^{i_1}_{-m_1}a^{j_1}_{-n_1}|0\rangle)_0 \dots U(a^{i_N}_{-m_N}a^{j_N}_{-n_N}|0\rangle)_0 |\chi\rangle , \tag{91}$$

using (10).

Let us restrict now to the case $d = 2$ for simplicity of notation. Relabel the oscillators a^1 and a^2 as a and b respectively. We take the ground state $|\chi\rangle$ to be a tensor product of the now known *one-dimensional* ground states for the one-dimensional subalgebras H_+ generated by the a and b oscillators (the representation of H^2_+ trivially decomposes into a sum of such representations). If we determine the matrix elements involving odd numbers of both a and b oscillators, then the matrix elements of the representation will be completely determined. In fact, clearly all that we have to do is to determine $\langle P | a_{-m}b_{-n}|0\rangle$ for all m, n (since any pair $U(a_{-m_1}b_{-n_1}|0\rangle)_0 U(a_{-m_2}b_{-n_2}|0\rangle)_0$ in (91) may be replaced by $U_0(a_{-m_1}b_{-n_1}|0\rangle * a_{-m_2}b_{-n_2}|0\rangle)_0$, which is even in both the a and b oscillators, and can be re-expressed in terms of $*$ products of states in the two copies of H_+).

Trivially we see that any quasi-primary state involving terms with one a oscillator and one b oscillator must be of the form

$$\alpha(\sum_p A_p a_{-p}b_{-N+p}|0\rangle , \tag{92}$$

(plus of course terms of other forms), where

$$pA_p + (N + 1 - p)A_{p-1} = 0. \tag{93}$$

Thus, from (31), we see that $\langle P|a_{-m}b_{-n}|0\rangle$ is determined in terms of $\langle P|a_{-1}b_{-N}|0\rangle$, $N < m + n$. We set $\langle P|a_{-1}b_{-N}|0\rangle = \rho_N$.

Now

$$\langle\chi|[(U(a_{-1}a_{-1}|0\rangle)_0, U(a_{-1}b_{-N})_0]|\chi\rangle = 0, \tag{94}$$

since we have chosen $U(a_{-1}a_{-1}|0\rangle)_0|\chi\rangle = 2\gamma^a|\chi\rangle$ for some scalar γ^a, corresponding to the conformal weight of the ground state with respect to the copy H_+^a of H_+ corresponding to a. But from (5), we find

$$[U(a_{-1}a_{-1}|0\rangle)_0, U(a_{-1}b_{-N})_0] = U(a_{-1}b_{-N}|0\rangle)_0 + U(a_{-2}b_{-N}|0\rangle)_0. \tag{95}$$

Then (94) gives

$$\langle\chi|U(a_{-2}b_{-N}|0\rangle)_0|\chi\rangle = -\rho_N. \tag{96}$$

We see from this, together with (31), that all of the ρ_N are determined in terms of ρ_1.

For example, the relevant term in the expansion (31) of $|P\rangle$ giving $a_{-1}b_{-2}|0\rangle$ is given by

$$\beta(2,1)\rho_1 L_{-1}a_{-1}b_{-1}|0\rangle = -\frac{1}{2}\rho_1(a_{-1}b_{-2} + a_{-2}b_{-1})|0\rangle, \tag{97}$$

and hence $\rho_2 = -\rho_1$.

At the next level we pick up a new quasi-primary state as in (92), and so for some $\alpha \in \mathbb{R}$ we find the two-oscillator odd b state in $|P\rangle$ to be

$$\beta(2,2)\rho_1 L_{-1}{}^2 a_{-1}b_{-1}|0\rangle \quad + \quad \alpha(a_{-1}b_{-3} - \frac{3}{2}a_{-2}b_{-2} + a_{-3}b_{-1})|0\rangle =$$

$$\frac{3}{10}\rho_1(a_{-1}b_{-3} + a_{-2}b_{-2} + a_{-3}b_{-1})|0\rangle \quad + \quad \alpha(a_{-1}b_{-3} - \frac{3}{2}a_{-2}b_{-2} + a_{-3}b_{-1})|0\rangle. \tag{98}$$

Now, (96) for $N = 2$ gives

$$4\left(\frac{3}{10}\rho_1 - \frac{3}{2}\alpha\right) = -\rho_2. \tag{99}$$

Hence

$$\rho_3 \equiv 3\alpha + \frac{9}{10}\rho_1 = \rho_1. \tag{100}$$

Similarly for the higher order terms.

[Note that, if desired, we may obtain directly an expression for ρ_N by the following trick, which also illustrates the sort of manipulations required to evaluate matrix elements in H^2_+ in terms of the representation of $H_+{}^2$.

Consider

$$\langle\chi|U(b_{-1}b_{-1}|0\rangle)_0 U(a_{-1}b_{-N}|0\rangle)_0|\chi\rangle = 2\gamma^b\rho_N, \tag{101}$$

where γ^b is the conformal weight of the ground state with respect to the copy H_+^b of H_+ corresponding to b.

However, using (10), we may rewrite this matrix element as

$$\langle\chi|U(b_{-1}b_{-1}|0\rangle \quad * \quad a_{-1}b_{-N}|0\rangle)_0|\chi\rangle = \langle\chi|U\left(a_{-1}b_{-1}{}^2 b_{-N}|0\rangle\right)_0|\chi\rangle$$
$$+ 2N\langle\chi|U\left(a_{-1}b_{-N}|0\rangle + 2a_{-1}b_{-N-1}|0\rangle + a_{-1}b_{-N-2}|0\rangle\right)_0|\chi\rangle, \tag{102}$$

19

using the explicit form for the $*$ product given in (9).

But

$$a_{-1}b^2_{-1}b_{-N}|0\rangle = b_{-1}b_{-N}|0\rangle * a_{-1}b_{-1}|0\rangle - (-1)^{N-1}N \sum_{p=1}^{N+2} \binom{N+1}{p-1} a_{-1}b_{-p}|0\rangle$$

$$-\frac{1}{2}N(N+1)a_{-1}b_{-N}|0\rangle - N(N+1)a_{-1}b_{-N-1}|0\rangle - \frac{1}{2}N(N+1)a_{-1}b_{-N-2}|0\rangle . \quad (103)$$

The matrix element of the first term on the right hand side is determined in terms of the known representation for the H_+ corresponding to the b's and also ρ_1. Together then these expressions provide the desired solution for ρ_N in terms of ρ_m, $m < N$ (for $N \geq 4$).]

We label the possible ground states for the one-dimensional representation of the algebra H_+ corresponding to a as

$$\psi_1 = |0\rangle , \qquad \psi_2 = |\lambda\rangle , \qquad \psi_3 = a_{-1}|0\rangle$$
$$\psi_4 = \rho , \qquad \psi_5 = c_{-\frac{1}{2}}\rho , \qquad\qquad (104)$$

as in (58) (we have dropped the $\pm$ subscript on ψ_2, as well as the various phase factors required to ensure reality of the states, simply for ease of notation – note also that we explicitly take the momentum λ to be non-zero). Similarly we have states ψ_i, $1 \leq i \leq 5$, corresponding to the algebra for b. We may take χ to be given by one of the 15 possible inequivalent tensor products $\psi_i \otimes \phi_j$. Let us consider the various possibilities and identify which correspond to ground states.

We may evaluate the norm of the state $U(a_{-1}b_{-1}|0\rangle)_2|\chi\rangle$ for $\chi = \psi_3 \otimes \phi_3 \equiv a_{-1}b_{-1}|0\rangle$ using the known representation structure as in the above calculations, *i.e.*

$$||U(a_{-1}b_{-1}|0\rangle)_2|\chi\rangle||^2 = \langle\chi|U(a_{-1}b_{-1}|0\rangle)_{-2}U(a_{-1}b_{-1}|0\rangle)_2|\chi\rangle$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_{|z|>|w|} \frac{dz}{2\pi i} z^{-1}w^3 \langle\chi|U(a_{-1}b_{-1}|0\rangle, z)U(a_{-1}b_{-1}|0\rangle, w)|\chi\rangle$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_{|z|>|w|} \frac{dz}{2\pi i} z^{-1}w^3 \sum_n \langle\chi|U(V(a_{-1}b_{-1})_{-n}a_{-1}b_{-1}|0\rangle, w)|\chi\rangle(z-w)^{n-2}$$

$$= \oint_0 \frac{dw}{2\pi i} \oint_{|z|>|w|} \frac{dz}{2\pi i} z^{-1} \sum_n w^{-n+1} \langle\chi|U(V(a_{-1}b_{-1})_n a_{-1}b_{-1}|0\rangle)_0|\chi\rangle(z-w)^{n-2} ,$$
$$(105)$$

using (5).

The matrix element may now be evaluated in terms of the known representations χ gives of $H_+{}^2$. Note that we do not need to do this explicitly, for we may use the known results. We have $||U(a_{-1}b_{-1}|0\rangle)_2|\chi\rangle||^2 = 1$, and so we deduce that $\psi_3 \otimes \phi_3$ is not consistent as a ground state.

Similarly, we may eliminate $\chi = \psi_5 \otimes \phi_5$ and $\psi_3 \otimes \phi_2$.

Now consider the possible ground states $\chi = \psi_i \otimes \phi_j$ with $i = 4, 5$, $j = 1, 2, 3$. Intuitively, these will be non-meromorphic representations since they are twisted in one coordinate and untwisted in the other. Let us verify this in a particular case.

We consider $\chi = \psi_4 \otimes \phi_1 \equiv \rho \otimes |0\rangle$, and the matrix element

$$\langle\chi|U(a_{-1}b_{-1}|0\rangle, z)U(a_{-1}b_{-1}|0\rangle, w)|\chi\rangle . \qquad (106)$$

By (5) this is

$$\sum_n \langle \chi | U\left(V(a_{-1}b_{-1}|0\rangle)_{-n} a_{-1}b_{-1}|0\rangle, w\right) |\chi\rangle (z-w)^{n-2} . \tag{107}$$

Now, because of the vacuum in the a sector, this sector is trivial, and the only terms in the sum which contribute are

$$(z-w)^{-4} + \sum_{n \geq 0} \langle \rho | U(b_{-1}b_{-n-1}|0\rangle, w) |\rho\rangle (z-w)^{n-2} . \tag{108}$$

We then see from (52) and (55) that this can be evaluated as

$$(z-w)^{-4} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n+1} \frac{n+1}{n+2} w^{-n-2} (z-w)^{n-2} = \frac{1}{2} \frac{z+w}{\sqrt{zw}} (z-w)^{-4} , \tag{109}$$

as required.

We are left with possible inequivalent ground states $\chi = a_{-1}|0\rangle$, $|\lambda\rangle$, ρ and $c_{-\frac{1}{2}}\rho$ (λ is now two-dimensional, and possibly zero). As we have shown, the matrix elements of the corresponding representation of $H^2{}_+$ are determined in terms of the parameter $\rho_1 = \langle \chi | U(a_{-1}b_{-1}|0\rangle)_0 |\chi\rangle$.

Now, as above, we may evaluate $||U(a_{-1}b_{-1}|0\rangle)_0|\chi\rangle||^2$. For $\chi = \rho$, this vanishes (note again that no calculation is necessary as we know that this is determined in terms of the known results), and so we have $\rho_1 = 0$ in this case and the representation is fixed uniquely.

For the other cases, we can in principle evaluate matrix elements of the form

$$\langle \chi | U(a_{-1}b_{-1}|0\rangle)_0 U(\psi)_0 U(a_{-1}b_{-1}|0\rangle)_0 |\chi\rangle \tag{110}$$

for $\psi \in H_+{}^2$. We will find that $U(a_{-1}b_{-1}|0\rangle)_0|\chi\rangle$ is a ground state for a representation of $H_+{}^2$ distinct from that corresponding to χ in the cases $\chi = a_{-1}|0\rangle$ and $\chi = c_{-\frac{1}{2}}\rho$. (For example, $V(a_{-1}b_{-1}|0\rangle)_0 a_{-1}|0\rangle = b_{-1}|0\rangle$.) We must therefore have $\langle \chi | U(a_{-1}b_{-1}|0\rangle)_0 |\chi\rangle = 0$, i.e. $\rho_1 = 0$.

The only possible ground state left to consider is $|\chi\rangle = |\lambda\rangle$. An explicit calculation must show that $U(a_{-1}b_{-1}|0\rangle)_0|\chi\rangle$ is a ground state for an isomorphic representation if $\gamma^a \gamma^b \neq 0$ (in fact $V(a_{-1}b_{-1}|0\rangle)_0|\lambda\rangle = \lambda^a \lambda^b |\lambda\rangle$).

We will obtain these same matrix elements if we take our representation to be given by

$$V\left(\prod_{i=1}^{M} a_{-m_i} \prod_{j=1}^{N} b_{-n_j}|0\rangle, z\right) =: \prod_{i=1}^{M} \frac{i}{(m_i-1)!} \frac{d^{m_i}}{dz^{m_i}} X^a(z) \prod_{j=1}^{N} \frac{i}{(n_j-1)!} \frac{d^{n_j}}{dz^{n_j}} X^b(z) : \Omega^N , \tag{111}$$

acting on a degenerate ground state consisting of copies $|\lambda\rangle_i$ of $|\lambda\rangle$, where X^a and X^b are the string fields (39) for a and b respectively and $\Omega^2 = 1$.

Ω is a hermitian matrix, since $\langle \lambda|_i U(a_{-1}b_{-1}|0\rangle)_0|\lambda\rangle_j{}^* = \langle \lambda|_j U(a_{-1}b_{-1}|0\rangle)_0|\lambda\rangle_i$ by the hermitian structure of the representation. Thus, diagonalising Ω gives the usual irreducible representations (the arbitrary sign ± 1 corresponding to the equivalence of representations given by the map θ_b ($\theta_b a_n \theta_b{}^{-1} = a_n$, $\theta_b b_n \theta_b{}^{-1} = -b_n$, $\theta_b a_n \theta_b{}^{-1} = a_n$, $\theta_a b_n \theta_a{}^{-1} = b_n$)).

The extension to more than two dimensions is similar. We start with a state χ which is a ground state for a representation of $H_+{}^d$. The matrix elements for arbitrary states in $H^d{}_+$ are then give as in (91).

We can argue, as before, that any matrix element involving $U(a^i_{-m}a^j_{-n}|0\rangle)_0$ $(i \neq j)$ can be reduced to ones involving just $U(a^i_{-1}a^j_{-1}|0\rangle)_0$. We can then go through arguments exactly as above to restrict the possible ground states χ by rejecting ones that lead to non-meromorphic matrix elements or states not annihilated by positive modes of the vertex operators defining the representation. The only problem we encounter in attempting to define the matrix elements for the representation completely is that we may have to evaluate matrix elements containing two or more such cross terms, *e.g.*

$$\langle\chi|\cdots U(a^i_{-1}a^j_{-1}|0\rangle)_0 U(a^k_{-1}a^l_{-1}|0\rangle)_0|\chi\rangle, \tag{112}$$

with i, j, k, l distinct. However, it is easy to verify (using only the results we know, since the calculation depends again only on the $H_+{}^d$ structure) that $||U(a^i_{-1}a^j_{-1}|0\rangle)_0 U(a^k_{-1}a^l_{-1}|0\rangle)_0|\chi\rangle||^2 = 0$ for all the allowable χ (and so we can evaluate everything in terms of scalars

$$\rho^{ij} \equiv \langle\chi|U(a^i_{-1}a^j_{-1}|0\rangle)_0|\chi\rangle \tag{113}$$

evaluated as above), except for $|\chi\rangle = |\lambda\rangle$ (and λ non-zero in all four relevant coordinates). We then have the situation as above, with the actions of the zero modes in this case given by matrices Ω_{ij} which commute for distinct indices (since the corresponding operators U_0 trivially commute from (5)). Simultaneously diagonalising them again gives the required result up to a trivial equivalence.

Finally, we must consider representations of $\mathcal{H}(\Lambda)_+$, where Λ is a non-zero even Euclidean lattice of dimension $d > 1$. The inclusion of momenta is essentially a straightforward extension of the above. Before so proceeding though, we must first argue that we can evaluate the coefficients of all terms in $|P\rangle$ given by the action of states in H^a_+ and H^b_+ (equivalently all zero momentum states with $\theta_a = \theta_b = 1$) on momentum states $|\lambda\rangle_+$, $\lambda \in 2\Lambda$. Note that again we restrict our considerations to two dimensions for simplicity of notation.

Suppose the representation is chosen so that L^a_0 and L^b_0 (using the obvious notation) act as scalars γ^a and γ^b respectively. As in (13) and (14), we see that for states ψ_a and ψ_b in H^a_+ and H^b_+ respectively

$$O(\psi_a, \psi_b) = (L^a_{-1} + L^a_0)\psi_a * \psi_b. \tag{114}$$

Now $\psi_a * \psi_b$ is trivially the state $\psi_a \otimes \psi_b$ in the full conformal field theory. We then deduce from (12) that we have, for terms in $|P\rangle$ given by the action of zero momentum states with $\theta_a = \theta_b = 1$ on $|\lambda\rangle_+$, $\lambda \in \Lambda$,

$$(L^a_1 + L^a_0)|P\rangle = (L^b_1 + L^b_0)|P\rangle = 0. \tag{115}$$

We then proceed exactly as before, and solve the separate one-dimensional problems for a and b (using L^a and L^b in place of L as appropriate) using our previous results (note that we need here the one-dimensional result with non-zero momentum). Finally, we use the orthogonality relation (33) with $a = |\lambda\rangle_+$, $\lambda \in \Lambda$. This clearly relates, as before, states in $|P\rangle$ with momentum $(n+1)\lambda$ to those with momentum $(n-1)\lambda$, and we are able to find all terms in $|P\rangle$ given by the action of zero momentum states with $\theta_a = \theta_b = 1$ on momentum states $|\lambda\rangle_+$, $\lambda \in 2\Lambda$, in terms of the representations of H^a_+ and H^b_+. (In fact, the terms for other momenta are given in terms of the coefficient for states $|\lambda\rangle_+$ for λ taking values only in the cosets $\Lambda/2\Lambda$ – the similarity to the gamma matrix representations discussed in [8] is not accidental, as will soon become apparent).

Now, exactly as argued before in the zero momentum case, all the matrix elements for the representation of $\mathcal{H}(\Lambda)_+$ are known once we have the matrix elements

$$\langle|\chi|U(\psi_1)_0\cdots U(\psi_M)_0 \quad U(\phi_1)_0\cdots U(\phi_N)_0|\chi\rangle \tag{116}$$

$$\langle|\chi|U(\psi_1)_0\cdots U(\psi_M)_0 \quad U(\phi_1)_0\cdots U(\phi_N)_0U(a_{-1}b_{-1}|0\rangle)_0|\chi\rangle \tag{117}$$

$$\langle|\chi|U(\psi_1)_0\cdots U(\psi_M)_0 \quad U(\phi_1)_0\cdots U(\phi_N)_0U(|\lambda\rangle_+)_0|\chi\rangle \tag{118}$$

$$\langle|\chi|U(\psi_1)_0\cdots U(\psi_M)_0 \quad U(\phi_1)_0\cdots U(\phi_N)_0U(a_{-1}b_{-1}|0\rangle)_0U(|\lambda\rangle_+)_0|\chi\rangle\,, \tag{119}$$

where $\psi_i \in H_+^a$, $\phi_j \in H_+^b$, $\lambda \in \Lambda$.

We start with χ a product of representations for H_+^a and H_+^b as before, and follow essentially the same argument. The possible ground states are as we found before. [Note that we can now say slightly more in the case of a representation with ground state $|\mu\rangle$. We find $||U(|\lambda\rangle_+)_n|\mu\rangle||^2 \neq 0$ for some $n > 0$ if μ is not of minimal norm in the set $\mu+\Lambda$ (we may calculate this norm using the structure of $|P\rangle$ which we know so far). That we also require $\mu \in \Lambda^*$ is easy to deduce by evaluating the four-point function as in (89).]

It is now simply a question of considering the separate possibilities as we did before. Since (116) and (117) are known, we need only evaluate (118) and (119).

First consider the ground state $a_{-1}|0\rangle$. Note that $||U(|\lambda\rangle_+)_0a_{-1}|0\rangle||^2 = 0$, except for $\lambda^2 = 2$ (we know this from the expected structure, and have enough of $|P\rangle$ to calculate it uniquely – hence again no calculation is necessary). For $\lambda^2 = 2$, we know that we may pick up (for $\lambda^a \neq 0$) the weight one state $|\lambda\rangle_+$. We can see this at our level of knowledge here by calculating matrix elements of the form

$$\langle\chi|U(|\lambda\rangle_+)_0U(\psi_1)_0\cdots U(\psi_M)_0U(\phi_1)_0\cdots U(\phi_N)_0U(|\lambda\rangle_+)_0|\chi\rangle\,. \tag{120}$$

This will show that $U(|\lambda\rangle_+)_0a_{-1}|0\rangle$ gives a representation of H_+^a, H_+^b distinct from that corresponding to $a_{-1}|0\rangle$, and we deduce that, in this case, (118) vanishes. Similarly, (119) also vanishes.

The argument for a ground state $|\mu\rangle$ is also straightforward. We know that $V(|\lambda\rangle_+)_0|\mu\rangle_+$ contains terms of momentum $\pm\mu \pm \lambda$. Thus again $U(|\lambda\rangle_+)_0|\mu\rangle_+$ can be seen to give rise to representations of H_+^a and H_+^b distinct from the original, except in the case $\lambda = \pm2\mu$. Thus, we can argue that, for $\lambda \neq \pm2\mu$, (118) and (119) vanish. The undetermined scalars $\langle\chi|U(|2\mu\rangle_+)_0|\chi\rangle$ and $\langle\chi|U(a_{-1}b_{-1}|0\rangle)_0U(|2\mu\rangle_+)_0|\chi\rangle$ can be easily calculated by requiring the representation to be consistent (*i.e.* requiring the operator product expansion of $U(|2\mu\rangle_+, z)$ with itself to be what it should be from (5)), or we can simply restrict to the case of a self-dual lattice $\Lambda^* = \Lambda$ (in which case $\mu \in \Lambda$ and $\lambda \in 2\Lambda$, and (33) with $a = |\lambda\rangle_+$ fixes the scalars in terms of the known zero-momentum states in $|P\rangle$).

For a twisted ground state ρ, $U(a_{-1}b_{-1}|0\rangle)_0U(|\lambda\rangle_+)_0\rho = 0$ (as we can see by evaluating the norm), but $U(|\lambda\rangle_+)_0\rho$ gives us a ground state of a representation which turns out to be identical to that generated from ρ. We thus find vertex operators as in (53), but with some matrix, say M_λ, in place of the gamma matrix γ_λ. Note that, by the comments at the end of the paragraph preceding (116), M_λ is only arbitrary for one state from each coset $\Lambda/2\Lambda$. Locality of the vertex operators (the analog of (2) for the U's – see [7]) then implies that

$$M_\lambda M_\mu = (-1)^{\lambda\cdot\mu}M_\mu M_\lambda\,, \tag{121}$$

and we recover the usual gamma matrices.

Finally, we consider a ground state $c_{-\frac{1}{2}}\rho$. We know that

$$V_T(|\lambda\rangle_+)_0 c_{-\frac{1}{2}}\hat{\rho} = 2^{-\lambda^2}\left((1 - 2\lambda^{1^2})c_{-\frac{1}{2}} - 2\lambda^1\lambda^2 d_{-\frac{1}{2}}\right)\gamma_\lambda\hat{\rho}\,, \tag{122}$$

for some spinor ground state $\hat{\rho}$. Then

$$R(\lambda)_0 c_{-\frac{1}{2}}\hat{\rho} \equiv 2^{-3}\left(3^2 2^{\lambda^2}U(|\lambda\rangle_+)_0 - 2^{9\lambda^2}U(|3\lambda\rangle_+)_0\right)c_{-\frac{1}{2}}\hat{\rho} = c_{-\frac{1}{2}}\hat{\rho}\,, \tag{123}$$

and similarly

$$U(a_{-1}b_{-1}|0\rangle)_0 R(\lambda)_0 c_{-\frac{1}{2}}\hat{\rho} = d_{-\frac{1}{2}}\hat{\rho}\,, \tag{124}$$

where d is the other twisted oscillator. Thus, using this combination in place of $U(|\lambda\rangle_+)_0$ (and remembering the comment at the end of the paragraph preceding (116)), we find that the modified (119) vanishes and (118) gives rises to again what turns out to be the gamma matrices.

This completes the sketch of the argument that the representations of $\mathcal{H}(\Lambda)_+$ comprise only the known untwisted and $\mathbb{Z}_2$-twisted representations detailed earlier in this paper.

Let us conclude this section with a simple observation. For a representation of $\mathcal{H}(\Lambda)_+$ with ground state $|\mu\rangle_\pm$, $\mu \in \Lambda^*$, the corresponding P has terms in $|2\mu\rangle_+$. However, these can only contribute to matrix elements if $2\mu \in \Lambda$, and further we see that we must have $2\mu^2$ even (so that this is a quasi-primary state of even weight in P) – in other words, matrix elements with the corresponding terms in P must vanish otherwise, and we can consistently set them to zero, as required by the equations satisfied by P. Thus note that if all allowed representations of $\mathcal{H}(\Lambda)_+$ have all terms in P not excluded from contributing to matrix elements by this argument, this is the same as saying $\sqrt{2}\Lambda^*$ is even, *i.e.* which is the same as saying that the $\mathbb{Z}_2$-orbifold $\mathcal{H}(\Lambda)_+ \oplus \mathcal{H}_T(\Lambda)_+$ is consistent [8, 23]. Further pursuit of such a point of view may enable us to better understand this condition on the lattice. (Note that consistency of the orbifold theory in this notation is decided by consideration of matrix elements of the form

$$\langle P|V(\psi_1, z_1)\cdots V(\psi_N, z_N)|P\rangle\,.) \tag{125}$$

6 Conclusions

We have proven that the known representations of the reflection-twisted Heisenberg algebra comprise the complete set of modules, and shown that the same is true in first one dimension and then given an argument that it is true in general for the reflection-twisted FKS lattice conformal field theories. In particular, we have found an alternative derivation for the rather mysterious term $\Delta(z)$ which occurs in the twisted sector vertex operators (52), initially found in [8, 23] by a rather ad hoc correction of a normal ordering problem. We find it in the solutions P_b and P_c to our constraint equations (31), (66) and (33), though we would like to understand more clearly the relation of the form of Δ to the embedding of $O(H_+)$ in $O(H)$ from which the twisted representation solutions arise. Also, we wish to understand better the relation of our work to that of Zhu, as well as the work of Dong, Li and Mason [12] who generalize Zhu's theory to twisted modules.

It must be stressed that, though the uniqueness of the $\mathbb{Z}_2$-twisted representation is the main result of this paper, the method used is of significance and is applicable to many other cases. In particular, there is an obvious extension to higher order twisted modules [27, 14].

In this paper, we have used the known results for the structure of the twisted modules in several places. It would be interesting to see also what happens in cases other than the simple twisted cases where we have less prior knowledge of the representation theory.

Also one can attempt to use the techniques employed here to develop a systematic means of classifying representations of any given meromorphic conformal field theory. In addition, one may try to construct orbifolds of theories for which there is no obvious geometric interpretation in terms of the propagation of a string on a singular manifold, since our point of view provides abstract tools not constrained by any requirement for an explicit construction (see the comments at the end of [25]). The analysis of the construction of orbifolds, and in particular the concept of an inverse to such a construction, by such techniques may help to realize the ideals of treating the original conformal field theory and its orbifold on the same footing [24, 31]

Our method, in much the same way as that of Dong in the case of the Leech lattice [10], is untidy however in comparison with that of [25] (though it does have the advantage of revealing explicitly how the twisted structure arises). Though the argument of [25], as discussed in the introduction, can only be regarded as heuristic at present, it would therefore appear judicious to attempt to tighten it. The main problem was the potentially non-analytic behavior of the matrix elements for a representation of the FKS theory defined in terms of some P satisfying the constraint equations. However, because matrix elements involving pairs of vertex operators for states in the odd-parity sector of the FKS theory can be rewritten in terms of those of the reflection-twisted projection, we can really only expect at worst square root branch cuts in any correlation function. We would expect similar results for any extension of a representation of a conformal field theory to a larger one in which it is *finitely* embedded. The formalizing of this rough argument is work in progress. Similar ideas occur in [13], and further investigation is required to elucidate the links. Note though that in general the techniques developed in the present paper will be required to analyze the representations of an arbitrary meromorphic conformal field theory, *i.e.* when no embedding of the conformal field theory in a larger simpler theory is available.

7   Acknowledgments

References

[1] P. Bouwknegt, J. McCarthy, and K. Pilch. *BRST analysis of physical states for 2d gravity coupled to $c \leq 1$ matter.* Comm. Math. Phys. **145** (1992) 541.

[2] E. Corrigan and T.J. Hollowood. *A bosonic representation of the twisted string emission vertex*. Nucl. Phys. **B303** (1988) 135.

[3] E. Corrigan and T.J. Hollowood. *Comments on the algebra of straight, twisted and intertwining vertex operators*. Nucl. Phys. **B304** (1988) 77.

[4] L. Dixon, D. Friedan, E. Martinec, and S. Shenker. *The conformal field theory of orbifolds*. Nucl. Phys. **B282** (1987) 13.

[5] L. Dixon, J. Harvey, C. Vafa, and E. Witten. *Strings on orbifolds*. Nucl. Phys. **B261** (1985) 678.

[6] L. Dixon, J. Harvey, C. Vafa, and E. Witten. *Strings on orbifolds (II)*. Nucl. Phys. **B274** (1986) 285.

[7] L. Dolan, P. Goddard, and P. Montague. *Conformal field theories, representations and lattice constructions*. Accepted for publication in Communications in Mathematical Physics, hep-th/9410029, DAMTP-94-80.

[8] L. Dolan, P. Goddard, and P. Montague. *Conformal field theory of twisted vertex operators*. Nucl. Phys. **B338** (1990) 529.

[9] L. Dolan, P. Goddard, and P. Montague. *Conformal field theory, triality and the Monster group*. Phys. Lett. **B236** (1990) 165.

[10] C. Dong. *Representations of the moonshine module vertex operator algebra*. Cotemporary Math **175** (1994) 27–36.

[11] C. Dong. *Twisted modules for vertex algebras associated with even lattices*. J. Algebra **165** (1994) 91–112.

[12] C. Dong, H. Li, and G. Mason. *Twisted representations of vertex operator algebras*. Santa Cruz preprint, 1995.

[13] C. Dong and Z. Lin. *Induced modules for vertex operator algebras*. Santa Cruz preprint, hep-th/9412038, 1995.

[14] C. Dong and G. Mason. *On the construction of the moonshine module as a $\mathbf{Z}_p$-orbifold*. Cotemporary Math **175** (1994) 37–52.

[15] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Academic Press, New York, 1988.

[16] I.B. Frenkel and V.G. Kac. *Basic representations of affine Lie algebras and dual resonance models*. Invent. Math. **62** (1980) 23.

[17] P. Goddard. *Meromorphic conformal field theory*. In *Infinite dimensional Lie algebras and Lie groups: Proceedings of the CIRM-Luminy Conference*, page 556, Singapore, 1989. World Scientific.

[18] T.J. Hollowood. *Twisted Strings, Vertex Operators and Algebras*. PhD thesis, Durham University, 1988.

[19] V.G. Kac. *Infinite dimensional Lie algebras.* Cambridge University Press, Cambridge, 1990.

[20] V.G. Kac and A.K. Raina. *Highest weight representations of infinite dimensional Lie algebras*, volume 2 of *Adv. Ser. Math. Phys.* World Scientific, 1987.

[21] J. Lepowsky. *Calculus of twisted vertex operators.* Proc. Natl. Acad. Sci. USA **82** (1985) 8295.

[22] I.G. MacDonald. *Symmetric Functions and Hall Polynomials.* Oxford University Press, 1979. Page 64, Example 6.

[23] P.S. Montague. *Codes, Lattices and Conformal Field Theories.* PhD thesis, Cambridge University, 1991.

[24] P.S. Montague. *Discussion of self-dual $c = 24$ conformal field theories.* hep-th/9205067, DAMTP-92-35, 1992.

[25] P.S. Montague. *On representations of conformal field theories and the construction of orbifolds.* Submitted for publication to Letters in Mathematical Physics, hep-th/9507083, preprint ADP-95-39/M34, DAMTP-95-42, 1994.

[26] P.S. Montague. *Orbifold constructions and the classification of self-dual $c = 24$ conformal field theories.* Nucl. Phys. **B428** (1994) 233. hep-th/9403088.

[27] P.S. Montague. *Third and higher order NFPA twisted constructions of conformal field theories from lattices.* Nucl. Phys. **B441** (1995) 337–382. hep-th 9502138.

[28] A.N. Schellekens. *Classification of ten-dimensional heterotic strings.* Phys. Lett. **B277** (1992) 277.

[29] A.N. Schellekens. *Meromorphic $c = 24$ conformal field theories.* Comm. Math. Phys. **153** (1993) 159.

[30] G. Segal. *Unitary representations of some infinite-dimensional groups.* Comm. Math. Phys. **80** (1981) 301.

[31] M.P. Tuite. *On the relationship between Monstrous moonshine and the uniqueness of the Monster module.* hep-th/9305057, 1993.

[32] C. Vafa. *Modular invariance and discrete torsion on orbifolds.* Nucl. Phys. **B273** (1986) 592.

[33] M. Wakimoto and H. Yamada. *Irreducible decompositions of Fock representations of the virasoro algebra.* Lett. Math. Phys. **7** (1983) 513.

[34] Y. Zhu. *Vertex operator algebras, elliptic functions, and modular forms.* 1990.